# Bilkent University
# Econ 101 - Fall 2023
# Chapter 2: Consumer Theory

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September 25, 2023

## Contents

# 1 A Very Brief Introduction

This lecture introduces the first formal model of the decision made by an economic agent: a **consumer**. We consider the simplest (and possibly the most widespread) form of economic interaction: the decision to buy some goods and services. This is literally the model of a consumer who is deciding to buy some goods in a supermarket. There are $n$ different goods, which have their own prices. The consumer observes the prices and decides how much to buy from each good.

As we discussed in the previous lecture, an economic agent has (i) a constraint and (ii) preferences. This economic agent is no different: the consumer has limited income, denoted by a number $I$. This is the consumer's constraint: she cannot buy everything she wants. The consumer also has well-defined preferences towards goods. We will discuss what "well-defined" means in a few pages.

Also, as discussed in the previous lecture, each economic model involves some simplifications. Here are a couple of simplifications we assume throughout this document:

- This is a one-time interaction.

  This means there is no room for "future" in this model: the customer does not go to the store and say "let me buy this item later". There may be future considerations in her decision to buy some goods (i.e., if she is buying a washing machine she realizes that she will most likely not need a washing machine in the near future). Those are fine – the future considerations, concerns about the quality of the good, uncertainty about its use, social concerns etc. are all captured by her preferences.

  Similarly, there is no "yesterday" either: the consumer does not say "I bought this item yesterday, so I do not need to buy it today". We allow the consumer to have such concerns (i.e., needing something because she did not buy it recently, or enjoying an item because she bought and enjoyed it in the past), but all these concerns are captured by the consumer's preferences.

- The customer observes all the prices perfectly, and knows her income. She can easily calculate the money required to buy a given group of items.

- The customer knows her preferences, and she acts according to her preferences. That is, she is **rational**.

At the risk of being repetitive: do these assumptions sometimes violated? Of course. But we are building a benchmark here.

# 2 The Model

There is a single consumer and $n$ different goods. The consumer decides how much to buy of each good, i.e. she chooses quantities.

Here is the **notation** we will use:

- $i$: Will be used to denote a generic **good**. We will use natural numbers to denote goods and to index them. Thus, the set of all goods will be denoted by $\{1, 2, \ldots, n\}$. Here $n$ denotes the $n$th good, and also the total number of goods that is available for consumption.

- $q_i$: Denotes the **quantity** of good $i$. The case where the consumer is considering consuming 5 kg of good 2 will be represented with $q_2 = 5$ kg. The quantity can be kilograms, grams, liters, numbers... Whatever the denomination is, we will say it is a **unit**. Note that for any good $i$ we must always have $q_i \geq 0$. $q_i = 0$ is allowed, i.e. the consumer may choose not to buy a good.

- $(q_1, q_2, \ldots, q_n)$: Denotes a consumption bundle (or simply a **bundle**). This is a list that represents how much of each good a consumer is considering for consumption. As an example, consider a situation where there are 4 possible goods that the consumer can consume (hence $n = 4$). The consumption bundle $(8, 2, 0, 12)$ represents the situation where the consumer is considering 8 units of good 1, 2 units of good 2, none of good 3, and 12 units of good 4 for consumption.

- $p_i$: Denotes the price of good $i$ per unit. Therefore, if the consumer buys $q_i$ units of good $i$ at price $p_i$, she pays $p_i q_i$ for that good.
- $I$: Denotes the income of the consumer (the total wealth of the consumer).

## 2.1 The Constraint

The constraint of the consumer specifies which bundles are affordable (i.e., feasible for the consumer) and which bundles are not.

**Definition 1.** *Given the prices $p_1, p_2, \ldots, p_n$ of the goods and the income $I$ of the consumer, a bundle $(q_1, q_2, \ldots, q_n)$ is* **feasible** *if and only if*

$$\sum_{i=1}^{n} p_i q_i = p_1 q_1 + p_2 q_2 + \cdots p_n q_n \leq I \quad .$$

*The set of feasible bundles is also called the* **budget set**. *The set of feasible bundles that requires the use of all the income, i.e., the bundles $(q_1, q_2, \ldots, q_n)$ such that $p_1 q_1 + p_2 q_2 + \cdots + p_n q_n = I$ are said to be on the* **budget line** *(they constitute the budget line).*

When there are two goods ($n = 2$), the set of feasible bundles given the prices $p_1, p_2$ and income $I$ are the bundles $(q_1, q_2)$ that satisfy:

$$p_1 q_1 + p_2 q_2 \leq I$$

They can be represented graphically with the orange region shown in Figure 1. The budget line is a line with slope $-\frac{p_1}{p_2}$. The (absolute value of) slope of the budget line is an important object we will revisit later. It is important because it is the answer to the following question: "if the consumer gives up one unit of good 1, how many units of good 2 she can buy?" Therefore, $\frac{p_1}{p_2}$ captures the rate at which the consumer can consume good 2 instead of consuming good 1. In other words, $\frac{p_1}{p_2}$ captures the *trade-off* faced by the consumer.

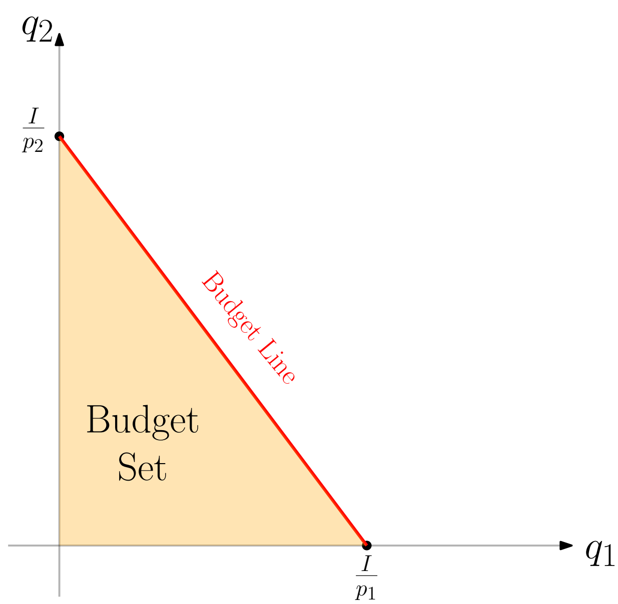


Figure 1: A graphical representation of feasible bundles given prices $p_1$, $p_2$, and income $I$.

## 2.2 Preferences

The preferences of the consumer specify the consumer's ranking between any two bundles $q = (q_1, q_2, \ldots, q_n)$ and $q' = (q'_1, q'_2, \ldots, q'_n)$. The consumer may strictly prefer one bundle over the other, or may feel indifferent between them. We capture the consumer's preferences through a preference relation.

A **preference relation** is a relation that compares two pairs of bundles. It is defined on **all** pairs of bundles, including (but not limited to) feasible bundles. Given two bundles $q$ and $q'$, it contains three possible scenarios.

- The situation where the consumer **(strictly) prefers** bundle $q = (q_1, q_2, \ldots, q_n)$ to another bundle $q' = (q'_1, q'_2, \ldots, q'_n)$ is represented by:

$$q \succ q'$$

- If the consumer is **indifferent** between the bundles $q$ and $q'$, then we will write:

$$q \sim q'$$

- If, for the consumer, bundle $q$ is **at least as good as** bundle $q'$ (i.e., the consumer prefers $q$ to $q'$ *or* she is indifferent between $q$ and $q'$) we will write:

$$q \succsim q'$$

  Therefore, for any pair of bundles $q$ and $q'$, we have:

$$q \succsim q' \iff (q \succ q' \text{ or } q \sim q') \tag{1}$$

  Note that when $q \succ q'$, by definition, it is also true that $q \succsim q'$. This is not really surprising: when a consumer **prefers** $q$ to $q'$, she also thinks $q$ is at least as good as $q'$.

  Similarly: when $q \sim q'$, by definition, it is also true that $q \succsim q'$. This is not surprising either: when a consumer **is indifferent between** $q$ to $q'$, she also thinks $q$ is at least as good as $q'$.

  What I want to point out that two of these scenarios can be satisfied at the same time. This is just like comparisons of real numbers. For the complete analogy: $\succ$ is like the $>$ sign, $\sim$ is like the $=$ sign, and $\succsim$ is like the $\geq$ sign. Now, as we know, $5 > 3$ and $5 \geq 3$ are both correct. Similarly, $4 = 4$ and $4 \geq 4$ are both correct.

A "well-defined" preference relation satisfies the following three conditions:

1. For any pair of bundles $q$ and $q'$,

$$q \succsim q' \quad \textit{or} \quad q' \succsim q. \tag{2}$$

   This condition states that the consumer is able compare any pair of bundles. A relation that satisfies this condition is said to be **complete**.

   Note that we could have equivalently stated the completeness condition as follows. For any pair of bundles $q$ and $q'$,

$$q \succ q' \quad \textit{or} \quad q \sim q' \quad \textit{or} \quad q' \succ q.$$

   (You can verify this by writing down equation (2) and using the definition of "at least as good as" relationship in equation (1).)

2. For any triple of bundles $q$, $q'$, and $q''$,

$$\textit{if } q \succsim q' \textit{ and } q' \succsim q'', \quad \textit{then } q \succsim q''.$$

   That is, for the consumer, if $q$ is at least as good as $q'$ and $q'$ is at least as good as $q''$, then $q$ should be at least as good as $q''$. A relation that satisfies this condition is said to be **transitive**.

There are several implications of transitivity condition, which you derive on your own. The first one is that if the consumer is indifferent between $q$ and $q'$ and if she is indifferent between $q'$ and $q''$, then she must be indifferent between $q$ and $q''$. In other words,

$$\textit{if } q \sim q' \textit{ and } q' \sim q'', \quad \textit{then } q \sim q''.$$

Another implication of transitivity is the following. If the consumer strictly prefers $q$ to $q'$ and if she is indifferent between $q'$ and $q''$, then she must strictly prefer $q$ to $q''$. In other words,

$$\textit{if } q \succ q' \textit{ and } q' \sim q'', \quad \textit{then } q \succ q''.$$

3. For any pair of bundles $q = (q_1, q_2, \ldots, q_n)$ and $q' = (q'_1, q'_2, \ldots, q'_n)$,

$$\textit{if } q_i < q'_i \textit{ for all } i \in \{1, 2, \ldots, n\}, \quad \textit{then } q' \succ q.$$

This condition states that if the bundle $q'$ has more of each good than bundle $q$ has, then $q'$ is preferred to $q$. To put it simply, *the consumer prefers more to less.* A relation that satisfies this condition is called **monotonic**.

Throughout this lecture, we assume that any preference relation is complete, transitive and monotonic, i.e., it is well-defined.

### 2.2.1 Indifference Curves

A graphical representation of preference relation can be obtained by drawing curves through bundles that the consumer is indifferent among. Figure 2 gives information about a preference relation in which the consumer is indifferent between the bundles $q$ and $q'$ (they are on the same curve).

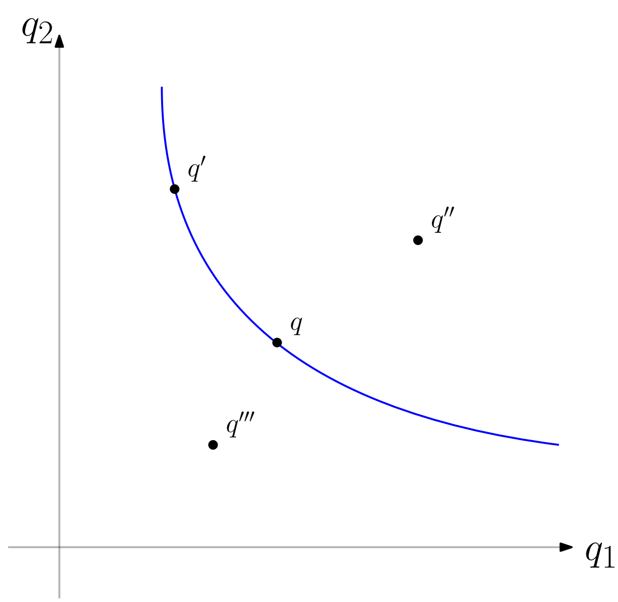


Figure 2: A graphical representation of a preference relation.

The curve that passes through $q$ is called **indifference curve** through $q$. Since $q''$ and $q'''$ are not on the indifference curve through $q$, the consumer is not indifferent between $q''$ and $q$ and also is not indifferent between $q'''$ and $q$. Since $q''$ contains more of the (two) goods than $q$ does, by the monotonicity of the preference relation, $q''$ is preferred to $q$. Similarly, since the bundle $q$ contains more of the goods than $q'''$ does, by the monotonicity of the preference relation, $q$ is preferred to $q'''$. Extending this argument, we can conclude that any bundle that lies "above" (in the north east side of) the indifference curve through $q$ is preferred to $q$ and $q$ is preferred to any bundle that lies "below" (in the south west side of) the indifference curve through $q$.

By now, you may have realized that it is possible to draw multiple indifference curves in the same graph. Consider, for instance, the indifference curve passing through $q''$. See Figure 3.

- By definition, the consumer is indifferent between any bundle on this "higher" indifference curve and $q''$. For instance, the consumer is indifferent between $q''''$ and $q''$:

$$q'''' \sim q''$$

- Recall that, by monotonicity, the consumer prefers $q''$ to $q$:

$$q'' \succ q$$

- Once again, by definition, the consumer is indifferent between any bundle on the "lower" indifference curve and $q$. For instance, the consumer is indifferent between $q'$ and $q$:

$$q \sim q'$$

- Combining all these stetements and using transitivity, we conclude:

$$q'''' \succ q'$$

Note that this argument can be repeated for any $q''''$ on the "higher" indifference curve and any $q'$ on the "lower" indifference curve. We conclude: *any bundle on a "higher" indifference curve is preferred over any bundle on a "lower" indifference curve.*

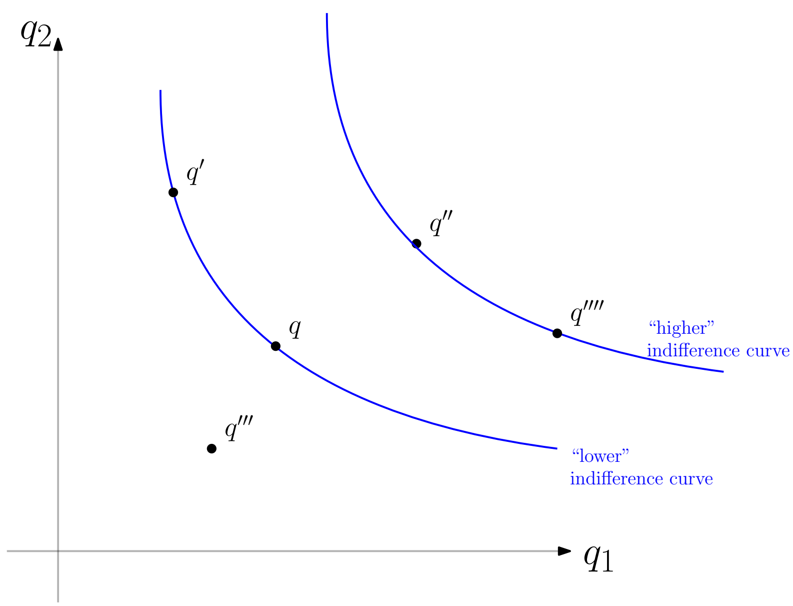


Figure 3: Multiple indifference curves. Any bundle on a "higher" indifference curve is preferred over any bundle on a "lower" indifference curve.

There are a couple of other things I want to emphasize about indifference curves:

- Since the preference relation of a consumer is assumed to be monotonic, the indifference curves must be downward sloping. Suppose, for a contradiction, that a part of an indifference is upward-sloping. Then, there are two bundles $q$ and $q'$ on the same indifference curve such that $q'_1 > q_1$ and $q'_2 > q_2$. By monotonicity, we must have $q' \succ q$. But then, $q$ and $q'$ cannot be on the same indifference curve! A contradiction.

- As long as transitivity and monotonicity are satisfied, indifference curves cannot cross. I am leaving this as an exercise for you to show.

### 2.2.2 Marginal Rate of Substitution

We will now define a very important object based on the indifference curves. It will provide the very crucial information on how much the consumer "values" good 1 over good 2 at a certain bundle.

The **marginal rate of substitution** of good 2 for good 1 at the bundle $q = (q_1, q_2)$, denoted $\text{MRS}_{2,1}(q)$, is the rate at which good 2 must substitute for a "small" decrease in the consumption of good 1 in order to keep the consumer indifferent to the initial bundle $q$. More formally (see Figure 4):

$$\text{MRS}_{2,1}(q_1, q_2) = \lim_{\substack{\Delta q_1 \to 0^+ \\ (q_1 - \Delta q_1, q_2 + \Delta q_2) \sim (q_1, q_2)}} \frac{\Delta q_2}{\Delta q_1} = |\text{slope of ind. curve at } q| \quad . \tag{3}$$

The marginal rate of substitution of good 1 for good 2 at the bundle $q$, denote $\text{MRS}_{1,2}(q)$, is similarly defined:

$$\text{MRS}_{1,2}(q_1, q_2) = \lim_{\substack{\Delta q_2 \to 0^+ \\ (q_1 + \Delta q_1, q_2 - \Delta q_2) \sim (q_1, q_2)}} \frac{\Delta q_1}{\Delta q_2} = \frac{1}{|\text{slope of ind. curve at } q|} \quad . \tag{4}$$

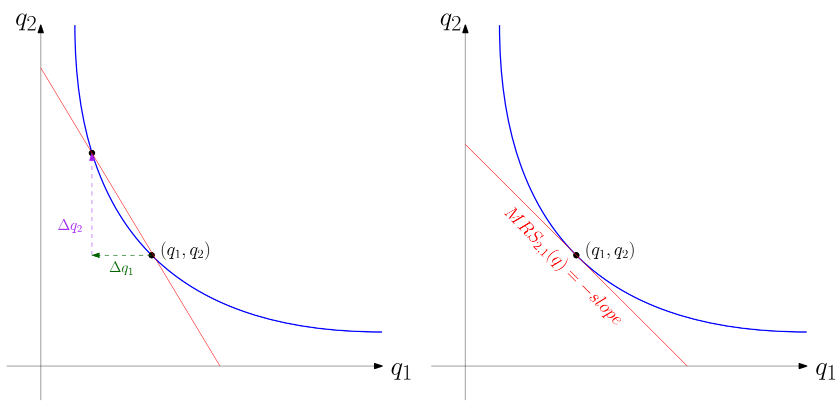


Figure 4: Marginal rate of substitution of good 2 for good 1: Taking the limit of $\Delta q_2/\Delta q_1$ as $\Delta q_1$ goes to zero in the figure on left we obtain the figure on the right.

$\text{MRS}_{2,1}(q)$ is a measure of how much the consumer "values" good 1 in terms of good 2 when he is endowed with the bundle $q$. Assume that the consumer is endowed with the bundle $q = (q_1, q_2)$. If we ask the consumer to give up a "small" amount of good 1, say $\Delta q_1$ units, the consumer would require "approximately" $\text{MRS}_{2,1}(q)\Delta q_1$ units of good 2, to compensate the reduction in the quantity of good 1 in order to be indifferent between the initial bundle and the bundle after the exchange. Similarly, if we asked to consumer to give up a "small" amount of good 2, say $\Delta q_2$ units, the consumer would require "approximately" $\text{MRS}_{1,2}(q)\Delta q_2$ units of good 1 to compensate the reduction in the quantity of good 2.

Generally speaking, there are four ways to interpret $\text{MRS}_{2,1}(q)$.

1. **(Mathematical.)** It is the limit of a ratio of two differences: see equation (3).
2. **(Verbal.)** It is a measure of how many units of good 2 the consumer must be given, so that she is left indifferent to a decrease in good 1.
3. **(Geometrical.)** It is the (absolute value of) the slope of the indifference curve: the steeper the indifference curve is, the higher $\text{MRS}_{2,1}(q)$ is.
4. **(Economical.)** It is a measure of the value of good 1 in terms of good 2: the more valuable good 1 is, the higher $\text{MRS}_{2,1}(q)$ is.

Let me now define two more properties on the preference, which are common features of many preferences in real life.

- Generally, when a consumer has more of a good (say, good 1), and less of another good (say, good 2), then good 1 becomes less "valuable" for the consumer relative to good 2. We formalize this idea as follows:

  Let $q$ and $q'$ be any two bundles that the consumer is indifferent between (i.e., they are on the same indifference curve). We require preference relations to be such that, if $q_1 > q_1'$, then $\text{MRS}_{2,1}(q) < \text{MRS}_{2,1}(q')$. Also, if $q_2 > q_2'$, then $\text{MRS}_{1,2}(q) < \text{MRS}_{1,2}(q')$.

  Preference relations that satisfy this condition are said to have **(strictly) diminishing marginal rate of substitution**. If a preference relation satisfies the diminishing marginal rate of substitution assumption, the relative value of good 1 in terms of good 2 will be lower as the consumer has more of good 1 and less of good 2. As a result, the indifference curve will get flatter as $q_1$ increases along the same indifference curve. This means: the indifference curves will be bowed toward the origin (Figure 5).

- A preference relation is said to be **smooth** if the indifference curves do not have any kinks. The first two indifference curves displayed in Figure 6 are examples of smooth preference relations and the next two are examples of preference relations that are not smooth. If a preference relation is smooth then for any bundle $q$ with positive component (i.e., $q_1 > 0$ and $q_2 > 0$) we have

$$\text{MRS}_{1,2}(q) = \frac{1}{\text{MRS}_{2,1}(q)} \quad . \tag{5}$$

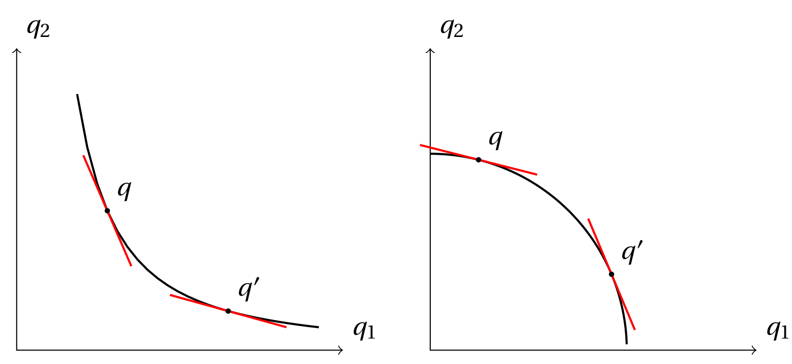


Figure 5: The graph one the left displays a indifference curve on which the diminishing marginal rate of substitution assumption holds. The graph on the right displays an indifference curve on which the marginal rate of substitution of good 2 for good 1 increases as we move in the increasing $q_1$ direction along the indifference curve. Hence the diminishing marginal rate of substitution assumption does not hold for the indifference curve on the right.

## 3 Optimal Bundle

Okay, now that we have a grasp of the consumer's constraints and preferences, it is time to characterize her choice. The following is a formal definition of the consumer's "favorite bundle among the feasible ones".

**Definition 2.** *Given the prices $p_1, p_2, \ldots, p_n$ and income $I$, a bundle $q^* = (q_1^*, q^*, \ldots, q_n^*)$ is an optimal bundle if and only if*

- $\sum_{i=1}^n p_i q_i^* = p_1 q_1^* + p_2 q_2^* + \cdots + p_n q_n^* \leq I$ *($q^*$ is feasible), and*
- *for any bundle $q = (q_1, q_2, \ldots, q_n)$, if $\sum_{i=1}^n p_i q_i \leq I$ (i.e., $q$ is feasible), then $q^* \succsim q$ (i.e., $q^*$ is at least as good as any feasible bundle).*

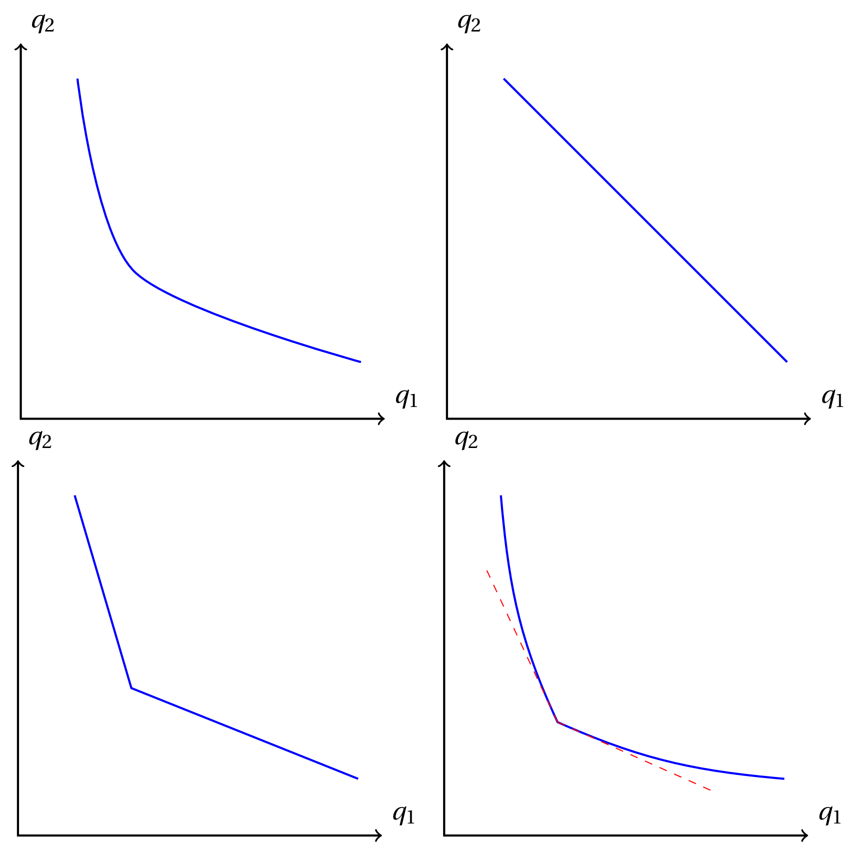


Figure 6: The indifference curves at the top are smooth and the indifference curves at the bottom have kinks.

Thus, a bundle is optimal if and only if it is feasible and is at least as good as (for the consumer) any feasible bundle. Alternatively, a bundle $q^*$ is optimal if and only if any bundle that is preferred to $q^*$ is not feasible.

We will now find the optimal bundle $q^*$ when there are two goods (the argument is generalizable to more than two goods). A starting point to develop the intuition is as follows. The optimality conditions imply that the consumer needs to find the **highest indifference curve, given her budget constraint**. Can you try to draw a budget set, the indifference curves, and find the optimal bundle?

Now, let's get more formal. Please note that all the statements below assume that the preferences are "well-defined" (i.e., they satisfy completeness, transitivity and monotonicity). In what follows, I will posit some claims – which are "Claims" in the mathematical sense, so they are correct statements under the assumptions we made. They are not "claims" in the colloquial sense. They have proofs. I am relegating the proofs to the Appendix to make this document more readable. Check them out if you are interested.

Assume that there are two goods and $q^* = (q_1^*, q_2^*) \in \mathbb{R}_+^2$ is an optimal bundle. Our first claim is that the **the optimal bundle must be on the budget line**.

**Claim 1.** *If $q^*$ is optimal, then $p_1 q_1^* + p_2 q_2^* = I$.*

Informally, Claim 1 means that the consumer must exhaust her budget under the optimal bundle. This is intuitively due to monotonocity: more is always better than less, and there is no reason to keep the money in the pocket, so you better just spend the money.

Our second claim is a subtle one that relates the marginal rate of substitution to the price ratio.

**Claim 2.** *If $q^*$ is optimal and $q_1^* > 0$, then*

$$\mathrm{MRS}_{2,1}(q^*) \geq \frac{p_1}{p_2} \quad . \tag{6}$$

Informally, Claim 2 says the following: if the consumer is buying good 1 in a strictly positive quantity, then it must be the case that she likes good 1 enough. Otherwise, buying good 1 would not be optimal.

The consumer could also consider consuming a "little" less of good 2 (provided that $q_2^* > 0$). Arguments similar to the above would yield the following claim.

**Claim 3.** *If $q^*$ is optimal and $q_2^* > 0$, then*

$$\mathrm{MRS}_{1,2}(q^*) \geq \frac{p_2}{p_1} \quad . \tag{7}$$

Informally, Claim 3 says: if the consumer is buying good 2 in a strictly positive quantity, then it must be the case that she likes good 2 enough. Otherwise, buying good 2 would not be optimal.

Now, it is time to combine everything we know and have the "big reveal" of consumer theory. That would be the theorem below.

**Theorem 1.** *Given the prices $p_1, p_2$ and income $I$, if $q^*$ is an optimal bundle, then $p_1 q_1^* + p_2 q_2^* = I$ and*

- *if $q_1^* > 0$, then*
$$\mathrm{MRS}_{2,1}(q^*) \geq \frac{p_1}{p_2} \quad ,$$
- *if $q_2^* > 0$, then*
$$\mathrm{MRS}_{1,2}(q^*) \geq \frac{p_2}{p_1} \quad ,$$
- *if $q_1^* > 0$, $q_2^* > 0$, and preference is smooth, then*
$$\mathrm{MRS}_{2,1}(q^*) = \frac{p_1}{p_2} \quad .$$

Figure 7 illustrates the optimal bundle when $q_1^* > 0$ and $q_2^* > 0$. Intuitively, it is the point where the indifference curve passing through $q^*$ *barely touches* the budget line, i.e. it is *tangent* to the budget line.

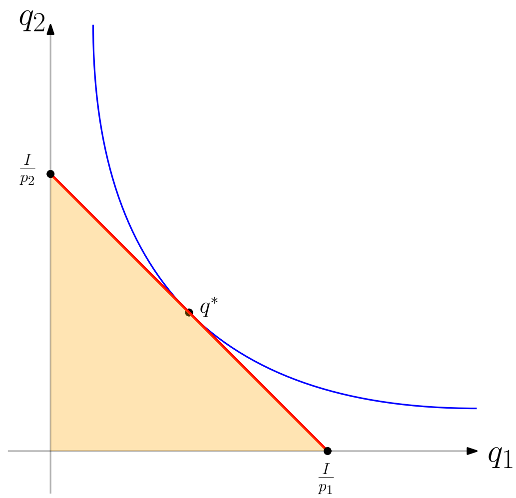


Figure 7: The optimal bundle $q^*$ is on the budget line and satisfies $\text{MRS}_{2,1}(q^*) = \frac{p_1}{p_2}$.

Take a moment to appreciate the beauty of this result! In the optimal bundle, the marginal rate of substitution is exactly equal to the price ratio. That is, suppose you go and ask the consumer in a supermarket:

> "I see your shopping cart, which contains your optimal bundle. Let me ask you a hypothetical question. At the optimal bundle, how much do you value good 1 in terms of good 2?"

And suppose her answer is:

> "Five. You need to give me five units of good 2 for me to give up one unit of good 1."

And then you go ask the cashier in the supermarket:

> "How expensive is good 1 relative to good 2?"

Almost magically, her answer is:

> "Five. You can give up five units of good 2 and buy one unit of good 1 instead."

Isn't this amazing? What is more amazing is that this applies to *every single consumer* in the supermarket, regardless of their preferences. Another customer may be a fan of good 1, i.e. her $\text{MRS}_{2,1}$ may be higher for every bundle. Fine, she keeps buying more of good 1 and less of good 2, until the marginal rate of substitution decreases and she finds the bundle where the marginal rate of substitution equals the price ratio.

In the appendix, I provide some examples of "famous" preferences and discuss the properties of the optimal bundle under those preferences. You may want to take a look at them before you solve some exercises.

# Appendix

## A Proofs

*Proof of Claim 1.* Suppose, towards a contradiction, that $p_1 q_1^* + p_2 q_2^* \neq I$. There are two possibilities.

- If $p_1 q_1^* + p_2 q_2^* > I$, $q^*$ would not be feasible. This would contradict feasibility of $q^*$.
- If $p_1 q_1^* + p_2 q_2^* < I$, the consumer can afford another bundle $q' = (q_1^* + \Delta_1, q_2^* + \Delta_2)$, i.e. $q'$ is feasible. Because the preference relation is monotonic, this is preferred to $q^*$, i.e. $q' \succ q^*$. This contradicts the optimality of $q^*$.

Since both cases lead to a contradiction, the proof follows. □

*Proof of Claim 2.* Assume that $q_1^* > 0$. Let $\Delta q_1$ be a "small" positive quantity such that $q_1^* - \Delta q_1 \geq 0$. (Since $q_1^*$ is positive, there is such a positive quantity).

The consumer is considering the bundle $q^*$. If she consumes $\Delta q_1$ units less of good 1 (i.e., consumes $q_1^* - \Delta q_1$ units of good 1 rather than $q_1^*$ units of it), then the consumer would require (approximately) $\text{MRS}_{2,1}(q^*)\Delta q_1$ units of good 2 to substitute for good 1 (so that the she is indifferent between the final bundle and the initial bundle $q^*$). But if the consumer buys $\Delta q_1$ units less of good 1, she would have $p_1 \Delta q_1$ TL to spend on good 2. With this money she can buy $(p_1 \Delta q_1)/p_2$ units of good 2. If

$$\text{MRS}_{2,1}(q^*)\Delta q_1 < \frac{p_1 \Delta q_1}{p_2} \quad , \tag{8}$$

then the consumer would become better off by consuming $\Delta q_1$ units less of good 1 and $(p_1 \Delta q_1)/p_2$ units more of good 2. That is, if (8) holds, then the bundle

$$\left(q_1^* - \Delta q_1,\ q^* + (p_1/p_2)\Delta q_1\right)$$

is feasible and is preferred to the bundle $q^*$. But this contradicts with $q^*$ being optimal. Thus, if $q^*$ is optimal, then (8) can not be true, which means that

$$\text{MRS}_{2,1}(q^*)\Delta q_1 \geq \frac{p_1 \Delta q_1}{p_2}$$

must be true. Since $\Delta q_1^*$ is positive, dividing both sides of the above inequality with $q_1^*$ we obtain:

$$\text{MRS}_{2,1}(q^*) \geq \frac{p_1}{p_2} \quad .$$

□

Proof of Claim 3 is very similar to that of Claim 2, so I am leaving it as an exercise.

*Proof of Theorem 1.* The proof of first two bullet points follow from Claims 2 and 3. For the last bullet point: If preference is smooth and $q^*$ is an optimal bundle with $q_1^* > 0$ and $q_2^* > 0$, then Claim 2, (5), and Claim 3 imply:

$$\frac{p_1}{p_2} \geq \frac{1}{\text{MRS}_{1,2}(q^*)} = \text{MRS}_{2,1}(q^*) \geq \frac{p_1}{p_2}$$

Which in turn implies

$$\text{MRS}_{2,1}(q^*) = \frac{p_1}{p_2} \quad .$$

□

# B Optimal Bundle for Some Famous Preferences

(This appendix is meant to be supplementary. It will hopefully serve as a guideline for future exercises. Please take some personal time to go through these examples on your own.)

The preferences we consider throughout this appendix satisfy completeness, transitivity and monotonicity. We will keep assuming that there are two goods for the sake of visualization, but once again the ideas extend. We will keep the budget constraint the same across examples: a bundle $q = (q_1, q_2)$ is feasible if and only if $p_1 q_1 + p_2 q_2 \leq I$.

## B.1 Perfect Substitutes

Suppose the consumer's preferences are such that: for any $q = (q_1, q_2)$ and $q' = (q'_1, q'_2)$,

$$q \succsim q' \iff aq_1 + bq_2 \geq aq'_1 + bq'_2 \tag{9}$$

where $a > 0$ and $b > 0$.

**How do indifference curves look like?** Recall that the consumer is indifferent between any two bundles on an indifference curve. Therefore, if $q$ and $q'$ are on the same indifference curve,

$$\begin{aligned} q \sim q' &\iff q \succsim q' \text{ and } q' \succsim q && \text{(by definition of indifference)} \\ &\iff aq_1 + bq_2 \geq aq'_1 + bq'_2 \text{ and } aq'_1 + bq'_2 \geq aq_1 + bq_2 && \text{(by the preferences in Equation (9))} \\ &\iff aq_1 + bq_2 = aq'_1 + bq'_2 \end{aligned}$$

What does it mean? Consider the line defined by the equation:

$$aq_1 + bq_2 = c \tag{10}$$

with some $c \geq 0$. The consumer is indifferent between any two bundles $q$ and $q'$ on this line, because

$$aq_1 + bq_2 = c = aq'_1 + bq'_2$$

Let's make sure that this is actually a line. Rearranging Equation (10), we arrive at:

$$q_2 = \frac{c}{b} - \frac{a}{b} q_1$$

which is, geometrically, the equation for a line with intercept $\frac{c}{b}$ and slope $-\frac{a}{b}$.

So the indifference curves in this case are **parallel lines**, each with slope $-\frac{a}{b}$. A higher $c$ means that the consumer is on a "higher" indifference curve, meaning that the consumer prefers the bundles on the indifference curves with higher $c$ to the bundles on the indifference curves with lower $c$. You can verify this using two alternative methods.

1. Take two indifference curves

   $$\begin{aligned} aq_1 + bq_2 &= c \\ aq_1 + bq_2 &= c' \end{aligned}$$

   with $c' > c$. Just drawing these indifference curves, you will see that the second indifference curve is higher than the first one (it is further away from the origin.)

   Take a bundle $q = (q_1, q_2)$ on the first indifference curve, and another bundle $q' = (q'_1, q'_2)$ on the second indifference curve. By the equations defining the indifference curves, they following equalities must hold:

   $$\begin{aligned} aq_1 + bq_2 &= c \\ aq'_1 + bq'_2 &= c' \end{aligned}$$

   But since $c' > c$, $aq'_1 + bq'_2 > aq_1 + bq_2$. Then, by the preferences in Equation (9), $q' \succ q$.

2. Just pick two bundles $q = (q_1, q_2)$ and $q' = (q'_1, q'_2)$ where $q'_1 > q_1$ and $q'_2 > q_2$. Draw the indifference curves that pass through $q$ and $q'$, and you will see that the one that passes through $q'$ is further away from the origin. By monotonicity,

$$q' \succ q$$

By the definition of an indifference curve, the consumer is indifferent between $q$ and any bundle $q''$ on the indifference curve passing through $q$.

$$q'' \sim q$$

Similarly, the consumer is indifferent between $q'$ and any bundle $q'''$ on the indifference curve passing through $q'$.

$$q''' \sim q'$$

By transitivity,

$$q''' \sim q' \succ q \sim q'' \implies q''' \succ q''$$

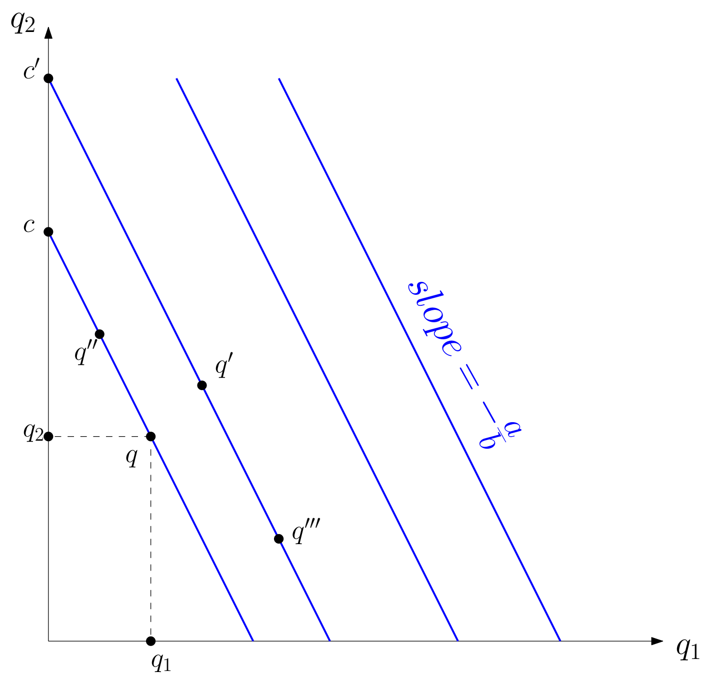


Figure 8: Indifference curves for preferences given in Equation (9).

**Fine, but what do they mean?** The interesting thing about lines is that their slopes are constant. Since the (absolute value of the) slope of the indifference curve is the marginal rate of substitution, it means that the marginal rate of substitution is constant.

$$MRS_{2,1}(q) = \frac{a}{b} \quad \text{for all } q$$

Recall that $MRS_{2,1}(q)$ is a measure of how much the consumer values good 1 in terms of good 2 when she is endowed with bundle $q$. When $MRS_{2,1}(q)$ is constant, this means that the relative value of good 1 does not depend on the bundle $q$. No matter how many units of good 1 and good 2 the consumer considers, the relative value is the same. **You can always take away $b$ units of good 1 from the consumer, compensate the consumer by giving $a$ extra units of good 2, and leave the consumer indifferent.** That is, no matter what the consumer is endowed with, $a$ units of good 2 can always **perfectly substitute** $b$ units of good 1. That's why good 1 and good 2 are **perfect substitutes**.

**Examples?** We typically consider the goods that are very similar in quality to be perfect substitutes. Take Coca Cola and Pepsi, for instance. You may like Coca Cola more than Pepsi, which is fine. In that the, the marginal rate of substitution of Pepsi for Coca Cola will be higher than one. What matters is that if you are willing to substitute one Coca Cola for one Pepsi when you have 10 Pepsis and 0 Coca Colas, then you should be willing to substitute one Coca Cola for one Pepsi when you have 9 Pepsis and 1 Coca Cola. So and so on.

**Is the diminishing marginal rate of substitution satisfied?** No. The marginal rate of substitution is constant.

**Are the preferences smooth?** Yes. The indifference curves do not have any kinks. Therefore, $MRS_{1,2}(q) = \frac{1}{MRS_{1,2}(q)} = \frac{1}{a/b} = \frac{b}{a}$ for all $q$.

**What is the optimal bundle?** Depends on $\frac{a}{b}$ (marginal rate of substitution) and $\frac{p_1}{p_2}$ (marginal rate of transformation.)

- To begin, suppose $\frac{a}{b} < \frac{p_1}{p_2}$ and consider the optimal bundle $q^* = (q_1^*, q_2^*)$. That is, **the indifference curves are flatter than the budget line**. I claim that in this case, we must have $q_1^* = 0$. Why? Suppose not, i.e., suppose $q_1^* > 0$. But then, by Theorem 1, we must have $MRS_{2,1}(q^*) \geq \frac{p_1}{p_2}$. But recall that $MRS_{2,1}(q^*) = \frac{a}{b} < \frac{p_1}{p_2}$. This is a contradiction. Therefore, we cannot have $q_1^* > 0$. We conclude that $q_1^* = 0$, and the consumer spends all her income on $q_2^*$. The optimal bundle is $q^* = (0, \frac{I}{p_2})$.

  Intuitively, what is going on? $\frac{a}{b}$ being low means that the consumer does not value good 1 much. Indeed, the relative price of good 1 in terms of good 2 is higher than the relative value of good 1 in terms of good 2. The consumer can always buy $b$ units less of good 1. This will save the consumer $bp_1$. With these savings, the consumer can buy an extra $\frac{bp_1}{p_2}$ units of good 2. Since $\frac{p_1}{p_2} > \frac{a}{b}$, $\frac{bp_1}{p_2} > a$. Thus, the consumer can buy *more than a* units of good 2 with her savings. But remember that $a$ units of good 2 would leave the consumer indifferent! Anything more than $a$ units of good 2 would make the consumer strictly happier! As a result, the consumer keeps reducing her consumption of good 1 until she has no good 1 left in her bundle.

  Geometrically, the following is going on:

  It's just the consumer finding the "highest" indifference curve subject to the budget constraint.

- Next, suppose $\frac{a}{b} > \frac{p_1}{p_2}$, i.e. **the indifference curves are steeper than the budget line**. Consider the optimal bundle $q^* = (q_1^*, q_2^*)$. I claim that in this case, we must have $q_2^* = 0$. Why? Suppose not, i.e., suppose $q_2^* > 0$. But then, by Theorem 1, we must have $MRS_{1,2}(q^*) \geq \frac{p_2}{p_1}$. But recall that $MRS_{1,2}(q^*) = \frac{b}{a} < \frac{p_2}{p_1}$. This is a contradiction. Therefore, we cannot have $q_2^* > 0$. We conclude that $q_2^* = 0$, and the consumer spends all her income on $q_1^*$. The optimal bundle is $q^* = (\frac{I}{p_1}, 0)$.

- Finally, consider the case $\frac{a}{b} = \frac{p_1}{p_2}$. The indifference curves are parallel to the budget line! In this case, there are **many** optimal bundles. Indeed, any bundle on the budget line is optimal.

  This is a somewhat knife-edge case (a consumer whose relative value exactly equals the relative price), but not impossible to find. For instance, suppose $a = b$ and $p_1 = p_2$. This means the consumer is totally indifferent between the goods (take away good 1, give good 2 in equal amounts, doesn't matter), and also the price are equal. This corresponds to cases where the brand of the item does not matter, at all. I am thinking of something like bleach. Does the brand of the bleach matter at all? For many people, no. When I need to buy bleach, I would just go ahead and buy the cheapest one. If the two brands in the supermarket have the same price, I could buy either of them.

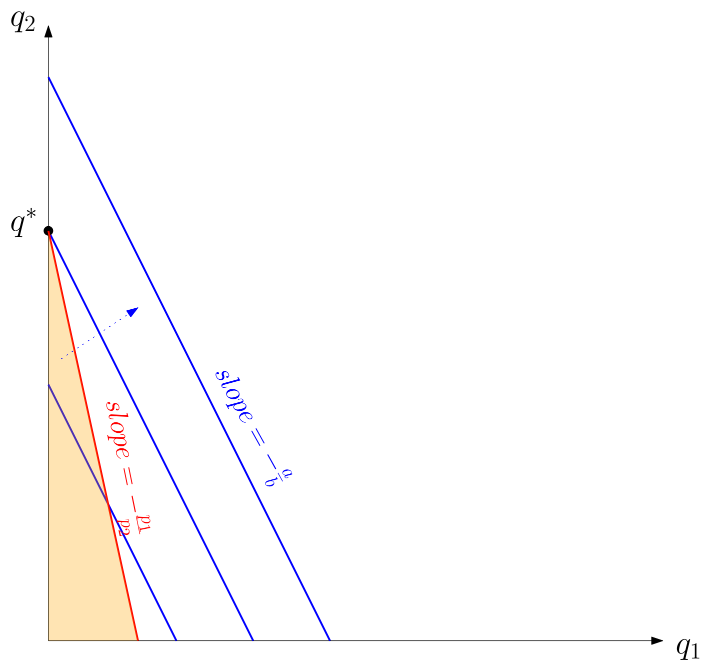


Figure 9: Optimal bundle when $\frac{a}{b} < \frac{p_1}{p_2}$. The blue lines are indifference curves and the red line is the budget line.

## B.2 Quasi-linear Preferences

Suppose the consumer's preferences are such that: for any $q = (q_1, q_2)$ and $q' = (q_1', q_2')$,

$$q \succsim q' \iff v(q_1) + q_2 \geq v(q_1') + q_2' \tag{11}$$

where $v(x)$ is an increasing and concave function. That is, its first derivative is positive and decreasing (it second derivative is negative). Think of $v(x) = \log x$ or $v(x) = x^\alpha$ for some $\alpha \in (0, 1)$.

**How do indifference curves look like?** If $q$ and $q'$ are on the same indifference curve,

$$\begin{aligned} q \sim q' &\iff q \succsim q' \text{ and } q' \succsim q && \text{(by definition of indifference)} \\ &\iff v(q_1) + q_2 \geq v(q_1') + q_2' \text{ and } v(q_1') + q_2' \geq v(q_1) + q_2 && \text{(by the preferences in Equation (11))} \\ &\iff v(q_1) + q_2 = v(q_1') + q_2' \end{aligned}$$

What does it mean? Consider the curve defined by the equation:

$$v(q_1) + q_2 = c \tag{12}$$

This is a typical indifference curve for quasi-linear preferences, where higher values of $c$ correspond to "higher" indifference curves. You can rearrange this to get:

$$q_2 = c - v(q_1) \tag{13}$$

Since $v(x)$ is increasing, this curve is downward-sloping. Since $v(x)$ is concave, this curve is convex. For a higher $c$, we shift this curve upwards.

**Fine, but what do they mean?** As you can guess by its name, quasi-linear preferences are "kind of" like linear preferences. By "kind of", we mean preferences are linear with respect to one good (in this case, good 2) and not linear with respect to the other good (good 1).

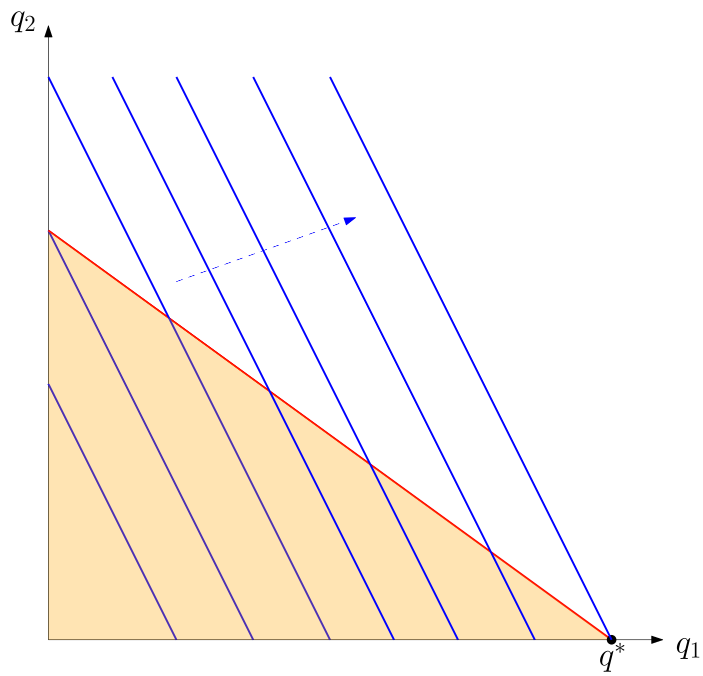


Figure 10: Optimal bundle when $\frac{a}{b} > \frac{p_1}{p_2}$. The blue lines are indifference curves and the red line is the budget line.

The marginal value that the consumer assigns to good 1 is decreasing in the amount of good 1 the consumer has. It is, however, constant in the amount of good 2 that the consumer has. When the consumer is endowed with more of good 1, she starts liking it less – this is like a usual good we consider. When the consumer is endowed with more of good 2, her attitudes towards good 2 does not change – this is like "linear" preferences.

You can see this feature by checking the marginal rate of substitution. Just take the derivative of Equation (13) and take its absolute value to find the slope of the indifference curve:

$$MRS_{2,1}(q) = v'(q_1) \quad \text{for all } q = (q_1, q_2)$$

As you see, this depends on $q_1$ but **not** on $q_2$. As long as $q_1$ remains the same, you can increase $q_2$ and $MRS_{2,1}(q)$ does not change. This means if you compare two indifference curves and keep $q_1$ constant, their slopes are the same. Therefore, indifference curves are just shifted versions of each other in the $y$-axis.

**Examples?** Good 1 in this example is a standard consumption good, like apples. Good 2 in this example is a good so that the consumer's feelings towards it does not change no matter how much of it she has. Let me give a somewhat radical example. Consider good 2 as **money**. The price of good 2 is $p_2 = 1$. That is, you can spend 1 TL and buy one unit of good 2, which is again 1 TL. Of course, this is just a representation: we are not considering a consumer who spends money to buy money. Instead, we are thinking about a consumer with a certain budget who decides how many apples to buy ($q_1$) and how much money to keep in her pocket ($q_2$). The crucial thing is that 1 TL is always 1 TL, no matter how much money you have. So it is reasonable to assume that consumers' feelings towards money does not depend on how much money they have already.[1]

[1]This does not have to be universally correct: you can imagine people valuing the extra lira less if they have more money already. That is, the preferences towards money can also satisfy diminishing marginal value. But especially for cash-constrained consumers, having quasi-linear preferences with respect to money seems like a reasonable thing to do.

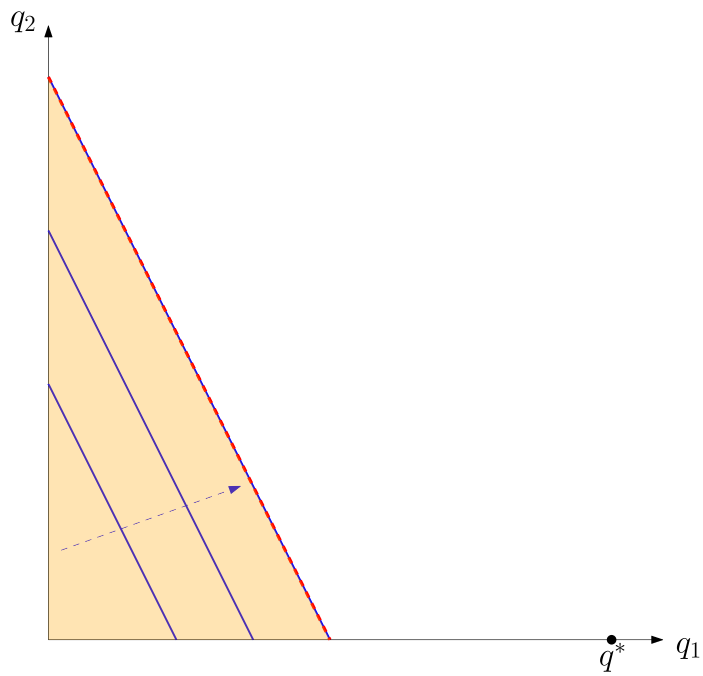


Figure 11: Optimal bundles when $\frac{a}{b} = \frac{p_1}{p_2}$.

I just want to point out: our earlier discussions made it look like the consumer has to spend all her income when she goes shopping. Now you realize that this framework allows for more general outcomes. It is possible to introduce money saved as another good and conduct the analysis as usual.

Another example: let $q_1$ denote the time spent on studying for the economics exam, and $q_2$ denote the time spent on other activities (such as watching more episodes of Ask-i Memnu). $v(q_1)$ is the expected grade on the economics exam when the student studies for $q_1$ hours. The student has a very standard thing to do when she does not study (the satisfaction you derive from Ask-i Memnu neither goes up nor goes down as you watch more of it), so the value of the alternative activities does not change at all.

**Is the diminishing marginal rate of substitution satisfied?** Yes. Recall that $v'(q_1)$ is decreasing.

**Are the preferences smooth?** Yes, as long as $v(x)$ is a smooth function (it does not have kinks).

**What is the optimal bundle?** The bottom line is that: the consumer keeps buying good 1 until the point where her marginal rate of substitution is low enough, so she does not want to buy it any more. The question is: what is the marginal rate of substitution if she, hypothetically, has spent all her income on good 1? At this point, she has bought $\frac{I}{p_1}$ units of good 1. If this much is enough so that $MRS_{2,1}(q)$ is lower than $\frac{p_1}{p_2}$, that is more than enough. She should have stopped buying earlier and spend the remaining amount on good 2. If $MRS_{2,1}(q) > \frac{p_1}{p_2}$ at this bundle, she spends all her income on good 1. If she had even more income she would buy even more of good 1, but she is constrained, so she stops here.

Formally, the optimal bundle depends on $v'(\frac{I}{p_1})$ (marginal rate of substitution) and $\frac{p_1}{p_2}$ (marginal rate of transformation.)

- If $v'(\frac{I}{p_1}) \leq \frac{p_1}{p_2}$, the optimal bundle is $q^* = (q_1^*, q_2^*)$ such that

$$MRS_{2,1}(q^*) = \frac{p_1}{p_2} \implies v'(q_1^*) = \frac{p_1}{p_2}$$

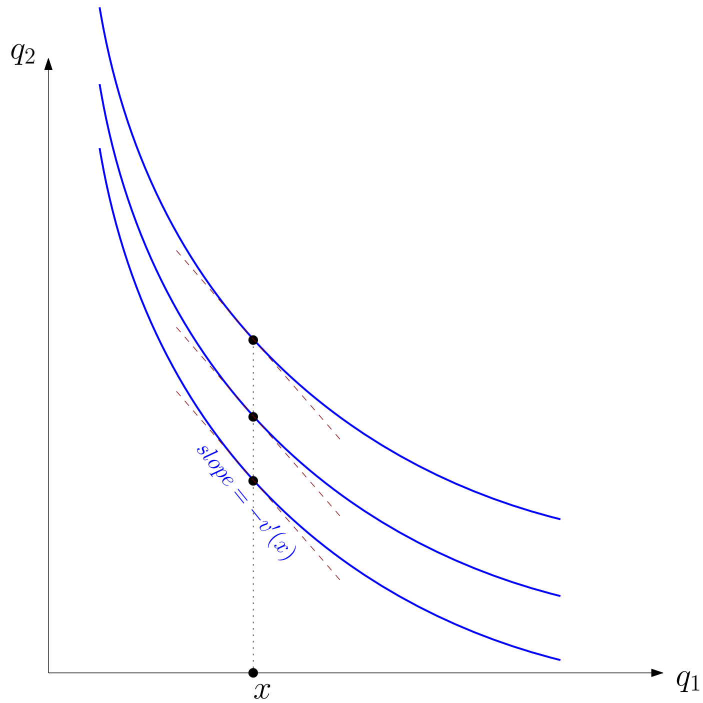


Figure 12: Indifference curves for preferences given in Equation (11).

and $q_2^* = \frac{I - p_1 q_1^*}{p_2}$ It's easier to see graphically; see Figure 13.

- If $v'(\frac{I}{p_1}) > \frac{p_1}{p_2}$, the optimal bundle is $q^* = (\frac{I}{p_1}, 0)$. See Figure 14.

## B.3 Cobb-Douglas Preferences

The following preferences are "invented" by Charles Cobb and Paul Douglas in the first half of 20th century.[2] Suppose the consumer's preferences are such that: for any $q = (q_1, q_2)$ and $q' = (q_1', q_2')$,

$$q \succsim q' \iff (q_1)^\alpha (q_2)^{1-\alpha} \geq (q_1')^\alpha (q_2')^{1-\alpha} \tag{14}$$

where $\alpha \in [0, 1]$ is a parameter that measures the "weight" that the consumer attaches to good 1 in her preferences. If $\alpha$ is higher, consumer has a higher inclination towards good 1.

**How do the indifference curves look like?** You can just imitate the arguments in the previous examples to derive the equation for a typical indifference curve:

$$(q_1)^\alpha (q_2)^{1-\alpha} = c \tag{15}$$

Once again, higher values of $c$ correspond to "higher" indifference curves. You can rearrange this to get:

$$q_2 = (c)^{\frac{1}{1-\alpha}} (q_1)^{\frac{-\alpha}{1-\alpha}} \tag{16}$$

You can check that this is downward-sloping.

[2]Fun fact: Paul Douglas later went on to serve as a senator ın the US for eighteen years! We, as economists, sometimes wish that he pushed for a legislation that requires every preference to be Cobb-Douglas. :)

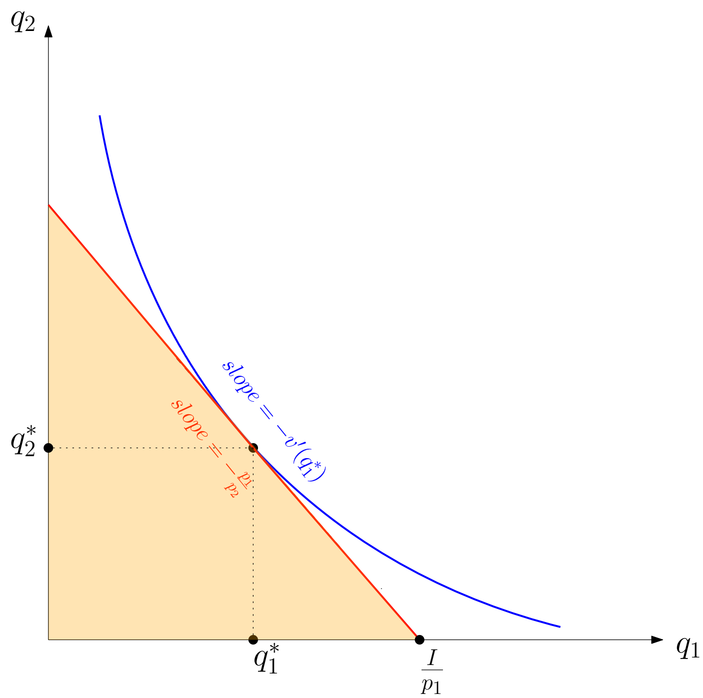


Figure 13: Optimal bundle when $v'(\frac{I}{p_1}) \leq \frac{p_1}{p_2}$.

**Fine, but what do they mean?** Not much in particular. Cobb-Douglas preferences are the standard preferences used to capture preferences towards two standard consumption goods. As you will see (and as we discussed in the lecture), these preferences satisfy all the nice properties of a typical preference relation. As you will also see, the optimal bundle satisfies certain nice properties.

With a little bit of messy algebra (which you don't need to know by heart), you can verify that:

$$MRS_{2,1}(q) = \frac{\alpha}{1-\alpha}\frac{q_2}{q_1} \quad \text{for all } q = (q_1, q_2) \tag{17}$$

So, the marginal rate of substitution depends **only** on the ratio of $q_2$ and $q_1$. Note that if $q_1$ decreases and $q_2$ increases, the marginal rate of substitution increases, i.e. good 1 becomes relatively more valuable to the consumer. This is the diminishing marginal rate of substitution!

Also note that the marginal rate of substitution is higher when $\alpha$ is higher, i.e., when the "weight" of good 1 is higher. This makes sense: if the weight of good 1 is higher, the consumer values good 1 more, which translates into a higher $MRS_{2,1}(q)$.

More importantly, as long as $\frac{q_2}{q_1}$ remains the same, the consumer's relative valuation of the good is the same. Suppose the goods are coffee and eggs. When the consumer is endowed with one cup of coffee and three eggs, she has a relative value attached to coffee versus eggs. If the consumer has Cobb-Douglas preferences, then she would have the same relative value when she has two cups of coffee and six eggs.

**Examples?** Usual consumption goods. Tea versus coffee. Apples versus bananas. White shirts versus blue shirts. Sujuk versus Halloumi cheese.

**Is the diminishing marginal rate of substitution satisfied?** Yessss.

**Are the preferences smooth?** Yessss.

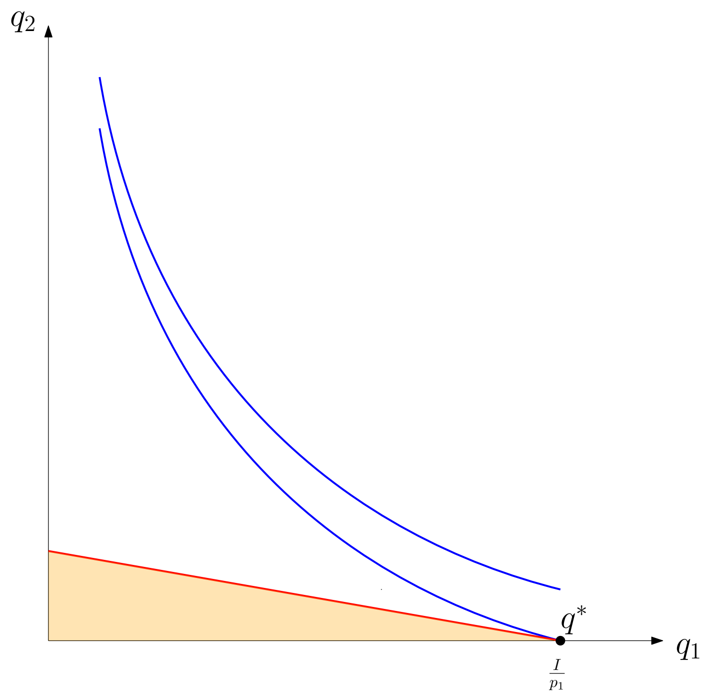


Figure 14: Optimal bundle when $v'(\frac{I}{p_1}) > \frac{p_1}{p_2}$. The consumer spends all her income on good 1.

**What is the optimal bundle?** Take my word for it when I say that in the optimal bundle $q^* = (q_1^*, q_2^*)$, the consumer has $q_1^* > 0$ and $q_2^* > 0$. Why? If $q_2^* = 0$ and $q_1^* > 0$, the consumer would have $MRS_{2,1}(q^*) = 0$. This is inconsistent with $q_1^* > 0$, as we showed in Theorem 1.

Given that $q_1^* > 0$ and $q_2^* > 0$, Theorem 1 yields:

$$MRS_{2,1}(q^*) = \frac{p_1}{p_2}$$

Use Equation (17) to substitute the left hand-side:

$$\frac{\alpha}{1-\alpha}\frac{q_2^*}{q_1^*} = \frac{p_1}{p_2}$$

Rearrange to get:

$$\frac{q_1^* p_1}{q_2^* p_2} = \frac{\alpha}{1-\alpha}$$

What does this mean? $q_1^* p_1$ is the consumer's total expenditure on good 1. $q_2^* p_2$ is the total expenditure on good 2. Combine this with the equation $q_1^* p_1 + q_2^* p_2 = I$ to derive the following result:

> "In the optimal bundle, the consumer spends $\alpha$ fraction of her income on good 1, and $1-\alpha$ fraction on good 2. Therefore, $q^* = (\frac{\alpha I}{p_1}, \frac{(1-\alpha)I}{p_2})$."

Circling back to what we had before: recall that $\alpha$ is the weight of good 1. If $\alpha$ is higher, the consumer allocates a larger share of her budget to good 1!

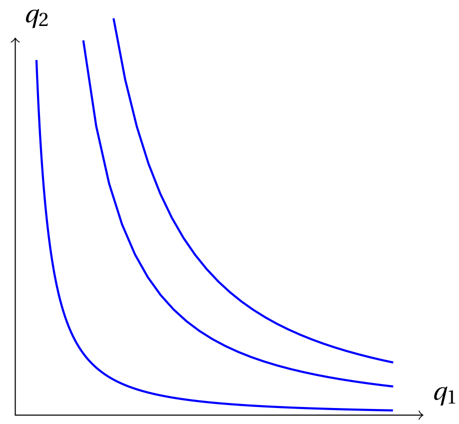


Figure 15: Indifference curves for preferences given in Equation (14). Note the interesting feature. If you draw a ray passing through the origin, this ray intersects each indifference curve once. Along the ray, $\frac{q_2}{q_1}$ is constant. Therefore, at those intersections, $MRS_{2,1}(q)$ will be the same.

## B.4 Perfect Complements

I will not write this one in much detail. For perfect complements, usually a visual inspection suffices.

Suppose the consumer's preferences are such that: for any $q = (q_1, q_2)$ and $q' = (q'_1, q'_2)$,

$$q \succsim q' \iff \min\{\frac{q_1}{a}, \frac{q_2}{b}\} \geq \min\{\frac{q'_1}{a}, \frac{q'_2}{b}\} \tag{18}$$

where $a > 0$ and $b > 0$.

A typical indifference curve is defined by the following equation:

$$\min\{\frac{q_1}{a}, \frac{q_2}{b}\} = c \tag{19}$$

Meaning? $a$ units of good 1 **perfectly complement** $b$ units of good 2. If the consumer has $a$ units of good 1 and more than $b$ units of good 2, the extra units of good 2 are useless. Examples? Left and right shoes, coffee and sugar (if you are drinking coffee only with sugar and if coffee is the only thing you put sugar in).

**Is the diminishing marginal rate of substitution satisfied?** Not really.

**Are the preferences smooth?** Nope.

**What is the optimal bundle?** See Figure 17. Theorem 1 does not apply in this case because the preferences are not smooth. But a visual inspection suffices.

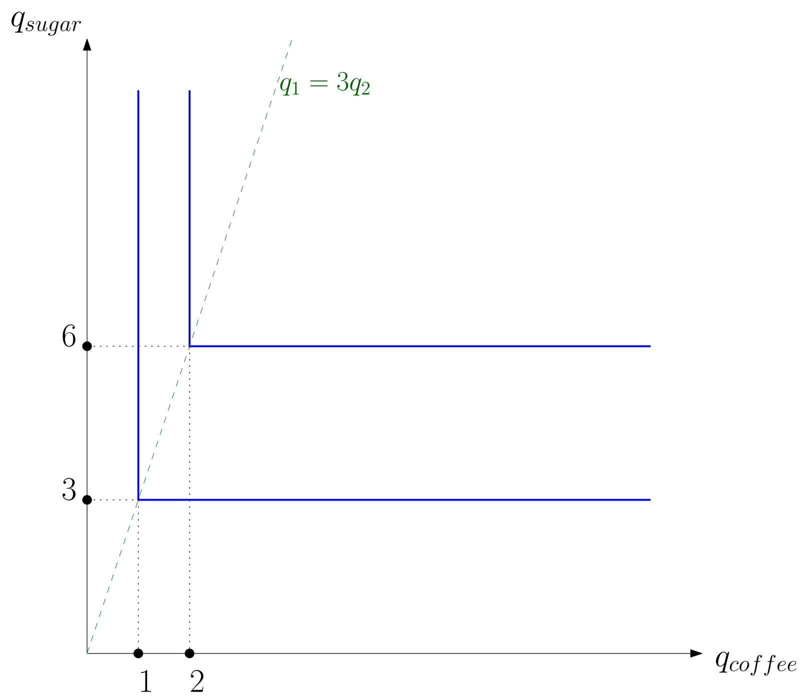


Figure 16: An example of indifference curves for preferences given in Equation (18). Here, good 1 is coffee (quantity is in cups) and good 2 is sugar (quantity is in cubes). The consumer consumes each cup of coffee with three cubes of sugar (I know – she should reduce her sugar consumption.) Therefore, one cup of coffee perfectly complements three cubes of sugar. Thus, $a = 1$ and $b = 3$.

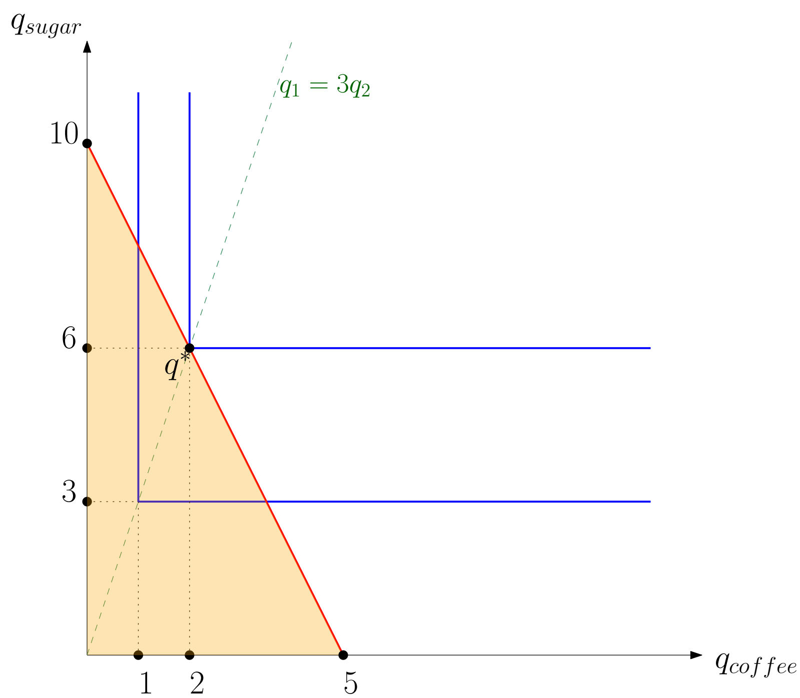


Figure 17: Optimal bundle. Suppose $p_1 = 5$ TL, $p_2 = 2.5$ TL and $I = 25$ TL. The consumer could buy 5 cups of coffee, but it will be worthless without the sugar. She could buy 10 cubes of sugar (how expensive is sugar, by the way???), but that would be worthless without the coffee. In the optimal bundle, she buys 2 cups of coffee with 6 sugars, which perfectly complement each other.