Search, Matching, and Signaling: Do Markets Help Relationships? Online Appendix

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Proof. **(of Lemma 4.)** The proof considers the two cases separately.

1. Assume

$$\frac{\overline{\mathcal{S}}(\mu^0) - \underline{\mathcal{S}}(\mu^0)}{(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}} \geq \frac{\underline{\mathcal{S}}(\mu^0)}{\mu^0 + (1-\mu^0)\frac{1-\delta}{1-\eta_0\delta}} \tag{1}$$

Two preliminary observations in this case are as follows.

Claim 1. $\delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi > \delta\pi \implies \delta\overline{\mathcal{S}}(\mu^0) > \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi.$

Proof. Assume $\delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi > \delta\pi$. Rearranging gives: $\pi < \frac{\underline{\mathcal{S}}(\mu^0)}{\mu^0+(1-\mu^0)\frac{1-\delta}{1-\eta_0\delta}}$. By Equation (1), this implies $\frac{\overline{\mathcal{S}}(\mu^0)-\underline{\mathcal{S}}(\mu^0)}{(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}} > \pi$. Rearranging gives: $\delta\overline{\mathcal{S}}(\mu^0) > \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi$. □

Claim 2. $\delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi > \delta\overline{\mathcal{S}}(\mu^0) \implies \delta\pi > \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi.$

Proof. Assume $\delta\underline{\mathcal{S}}(\mu^0)+\delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi > \delta\overline{\mathcal{S}}(\mu^0)$. Rearranging gives: $\pi > \frac{\overline{\mathcal{S}}(\mu^0)-\underline{\mathcal{S}}(\mu^0)}{(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}}$. By Equation (1), this implies $\pi > \frac{\underline{\mathcal{S}}(\mu^0)}{\mu^0+(1-\mu^0)\frac{q-\delta}{1-\eta_0\delta}}$. Rearranging gives: $\delta\pi > \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi$. □

Claim 1 and 2 combined implies $\delta\underline{\mathcal{S}}(\mu^0)+\delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi \leq \max\{\delta\pi, \delta\overline{\mathcal{S}}(\mu^0\}$. It follows that in this case, $\mathcal{V}_f(\mu^0,\pi) = \max\{\delta\pi, \delta\overline{\mathcal{S}}(\mu^0\}$.

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2. Assume

$$\frac{\overline{\mathcal{S}}(\mu^0) - \underline{\mathcal{S}}(\mu^0)}{(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}} < \frac{\underline{\mathcal{S}}(\mu^0)}{\mu^0 + (1-\mu^0)\frac{1-\delta}{1-\eta_0\delta}} \tag{2}$$

In this case,

(a) Suppose

$$\pi \leq \frac{\overline{\mathcal{S}}(\mu^0) - \underline{\mathcal{S}}(\mu^0)}{(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}} \tag{3}$$

By Equation (2), this implies: $\pi < \frac{\underline{\mathcal{S}}(\mu^0)}{\mu^0+(1-\mu^0)\frac{1-\delta}{1-\eta_0\delta}}$. Rearranging gives:

$$\delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta})\pi > \delta\pi \tag{4}$$

Also, rearranging Equation (3) gives:

$$\delta\overline{\mathcal{S}}(\mu^0) \geq \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi \tag{5}$$

Combining Equation (4) and (5), one concludes: $\mathcal{V}_f(\mu^0,\pi) = \delta\overline{\mathcal{S}}(\mu^0)$.

(b) Suppose

$$\pi \in [\frac{\overline{\mathcal{S}}(\mu^0) - \underline{\mathcal{S}}(\mu^0)}{(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}}, \frac{\underline{\mathcal{S}}(\mu^0)}{\mu^0 + (1-\mu^0)\frac{1-\delta}{1-\eta_0\delta}}] \tag{6}$$

Rearranging $\pi \geq \frac{\overline{\mathcal{S}}(\mu^0)-\underline{\mathcal{S}}(\mu^0)}{(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}}$ yields: $\delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi \geq \delta\overline{\mathcal{S}}(\mu^0)$. Rearranging $\pi \leq \frac{\underline{\mathcal{S}}(\mu^0)}{\mu^0+(1-\mu^0)\frac{1-\delta}{1-\eta_0\delta}}$ yields: $\delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi \geq \delta\pi$. Combining, one concludes: $\mathcal{V}_f(\mu^0,\pi) = \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi$.

(c) Suppose

$$\pi \geq \frac{\underline{\mathcal{S}}(\mu^0)}{\mu^0 + (1-\mu^0)\frac{1-\delta}{1-\eta_0\delta}} \tag{7}$$

By Equation (2), this implies: $\pi > \frac{\overline{\mathcal{S}}(\mu^0)-\underline{\mathcal{S}}(\mu^0)}{(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}}$. Rearranging gives:

$$\delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi > \delta\overline{\mathcal{S}}(\mu^0) \tag{8}$$

Also, rearranging Equation (7) gives:

$$\delta\pi \geq \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)\frac{\delta-\eta_0\delta}{1-\eta_0\delta}\pi \tag{9}$$

Combining Equation (8) and (9), one concludes: $\mathcal{V}_f(\mu^0,\pi) = \delta\pi$.

Since all the cases considered above are exhaustive, the proof follows. □

Proof. **(of Lemma 5.)** For expositional simplicity throughout this proof, let:

$$\alpha := \frac{1-\delta}{1-\eta_0\delta} \qquad 1-\alpha := \frac{\delta-\eta_0\delta}{1-\eta_0\delta}$$

The proof considers three cases.

1. Suppose $\mathcal{V}_f(\mu^0) = \delta\pi$, i.e. $\delta\pi \geq \max\{\delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi, \delta\overline{\mathcal{S}}(\mu^0)\}$.

   I will demonstrate that $k^*(\mu^0,\pi) = 0$, i.e. $\delta V_f(0,\mu^0,\pi) - c(0) \geq \delta V_f(k,\mu^0,\pi) - c(k)$ for all $k \geq 0$.

   Note that in this case, $\delta V_f(0,\mu^0,\pi) - c(0) = \delta\pi$, because:

$$0 \leq \frac{1-\delta}{1-\beta}\frac{1}{\gamma(\mu^0)}\pi$$

   and $c(0) = 0$ by Assumption 1.

   - For $k \in (0, \frac{1-\delta}{1-\beta}\frac{1}{\gamma(\mu^0)}\pi]$,

$$\begin{aligned}\delta V_f(0,\mu^0,\pi) - c(0) &= \delta\pi \\ &> \delta\pi - c(k) \\ &= \delta V_f(k,\mu^0,\pi) - c(k)\end{aligned}$$

     where the inequality follows by Assumption 1.

   - For $k \in \left[\frac{1-\delta}{1-\beta}\frac{1}{\gamma(\mu^0)}\pi, \frac{1-\delta}{1-\beta}\frac{1}{\eta_0}\pi\right]$,

$$\begin{aligned}\delta V_f(0,\mu^0,\pi) - c(0) &= \delta\pi \\ &\geq \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi \\ &= \delta\left(\frac{1}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\alpha\eta_0)\underline{k}(\mu^0) - \frac{c(\underline{k}(\mu^0))}{\delta}\right) + \delta(1-\mu^0)(1-\alpha)\pi \\ &\geq \delta\left(\frac{1}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\alpha\eta_0)k - \frac{c(k)}{\delta}\right) + \delta(1-\mu^0)(1-\alpha)\pi \\ &= \frac{\delta}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\alpha\eta_0)k + \delta(1-\mu^0)(1-\alpha)\pi - c(k) \\ &= \delta V_f(k,\mu^0,\pi) - c(k)\end{aligned}$$

     where the first inequality holds in the case considered here, and the second inequality follows by Equation (26).

- For $k \geq \frac{1-\delta}{1-\beta}\frac{1}{\eta_0}\pi$,

$$\begin{aligned}
\delta V_f(0, \mu^0, \pi) - c(0) &= \delta\pi \\
&\geq \delta\overline{\mathcal{S}}(\mu^0) \\
&= \delta\left(\frac{1}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\eta_0)\overline{k}(\mu^0) - \frac{c(\overline{k}(\mu^0))}{\delta}\right) \\
&\geq \delta\left(\frac{1}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\eta_0)k - \frac{c(k)}{\delta}\right) \\
&= \frac{\delta}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\eta_0)k - c(k) \\
&= \delta V_f(k, \mu^0, \pi) - c(k)
\end{aligned}$$

where the first inequality holds in the case considered here, and the second inequality follows by Equation (24).

2. Suppose $\mathcal{V}_f(\mu^0) = \delta\overline{\mathcal{S}}(\mu^0)$, i.e. $\delta\overline{\mathcal{S}}(\mu^0) \geq \max\{\delta\pi, \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi\}$.

I will demonstrate that $k^*(\mu^0, \pi) = \overline{k}(\mu^0)$, i.e. $\delta V_f(\overline{k}(\mu^0), \mu^0, \pi) - c(\overline{k}(\mu^0)) \geq \delta V_f(k, \mu^0, \pi) - c(k)$ for all $k \geq 0$.

My first claim is that in this case, $\delta V_f(\overline{k}(\mu^0), \mu^0, \pi) - c(\overline{k}(\mu^0)) = \frac{\delta}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\eta_0)\overline{k}(\mu^0) - c(\overline{k}(\mu^0))$. This is equivalent to showing that $\overline{k}(\mu^0) \geq \frac{1-\delta}{1-\beta}\frac{1}{\eta_0}\pi$. Suppose not, i.e. suppose $\pi > \frac{1-\beta}{1-\delta}\eta_0\overline{k}(\mu^0)$. But then,

$$\begin{aligned}
\delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi &= \frac{\delta}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\alpha\eta_0)\underline{k}(\mu^0) - c(\underline{k}(\mu^0)) + \delta(1-\mu^0)(1-\alpha)\pi \\
&> \frac{\delta}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\alpha\eta_0)\overline{k}(\mu^0) - c(\overline{k}(\mu^0)) + \delta(1-\mu^0)(1-\alpha)\pi \\
&\geq \frac{\delta}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\alpha\eta_0)\overline{k}(\mu^0) - c(\overline{k}(\mu^0)) + \delta(1-\mu^0)(1-\alpha)\frac{1-\beta}{1-\delta}\eta_0\overline{k}(\mu^0) \\
&= \frac{\delta}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\eta_0)\overline{k}(\mu^0) - c(\overline{k}(\mu^0)) \\
&= \delta\overline{\mathcal{S}}(\mu^0)
\end{aligned}$$

where the first inequality follows by Equation (26). This contradicts the case considered here. Therefore,

$$\delta V_f(\overline{k}(\mu^0), \mu^0, \pi) - c(\overline{k}(\mu^0)) = \frac{\delta}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\eta_0)\overline{k}(\mu^0) - c(\overline{k}(\mu^0)) = \delta\overline{\mathcal{S}}(\mu^0)$$

- For $k \in [0, \frac{1-\delta}{1-\beta}\frac{1}{\gamma(\mu^0)}\pi]$,

$$\begin{aligned}
\delta V_f(\overline{k}(\mu^0), \mu^0, \pi) - c(\overline{k}(\mu^0)) &= \delta\overline{\mathcal{S}}(\mu^0) \\
&\geq \delta\pi \\
&\geq \delta\pi - c(k) \\
&= \delta V_f(k, \mu^0, \pi) - c(k)
\end{aligned}$$

where the first inequality holds in the case considered here, and the second inequality holds by Assumption 1.

- For $k \in \left[\frac{1-\delta}{1-\beta}\frac{1}{\gamma(\mu^0)}\pi, \frac{1-\delta}{1-\beta}\frac{1}{\eta_0}\pi\right]$,

$$\begin{aligned}
\delta V_f(\overline{k}(\mu^0),\mu^0,\pi) - c(\overline{k}(\mu^0)) &= \delta\overline{\mathcal{S}}(\mu^0) \\
&\geq \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi \\
&= \delta\left(\frac{1}{1-\delta}(1-\beta)(\mu^0+(1-\mu^0)\alpha\eta_0)\underline{k}(\mu^0) - \frac{c(\underline{k}(\mu^0))}{\delta}\right) + \delta(1-\mu^0)(1-\alpha)\pi \\
&\geq \delta\left(\frac{1}{1-\delta}(1-\beta)(\mu^0+(1-\mu^0)\alpha\eta_0)k - \frac{c(k)}{\delta}\right) + \delta(1-\mu^0)(1-\alpha)\pi \\
&= \frac{\delta}{1-\delta}(1-\beta)(\mu^0+(1-\mu^0)\alpha\eta_0)k + \delta(1-\mu^0)(1-\alpha)\pi - c(k) \\
&= \delta V_f(k,\mu^0,\pi) - c(k)
\end{aligned}$$

where the first inequality holds in the case considered here, and the second inequality follows by Equation (26).

- For $k \geq \frac{1-\delta}{1-\beta}\frac{1}{\eta_0}\pi$,

$$\begin{aligned}
\delta V_f(\overline{k}(\mu^0),\mu^0,\pi) - c(\overline{k}(\mu^0)) &= \delta\overline{\mathcal{S}}(\mu^0) \\
&= \delta\left(\frac{1}{1-\delta}(1-\beta)(\mu^0+(1-\mu^0)\eta_0)\overline{k}(\mu^0) - \frac{c(\overline{k}(\mu^0))}{\delta}\right) \\
&\geq \delta\left(\frac{1}{1-\delta}(1-\beta)(\mu^0+(1-\mu^0)\eta_0)k - \frac{c(k)}{\delta}\right) \\
&= \frac{\delta}{1-\delta}(1-\beta)(\mu^0+(1-\mu^0)\eta_0)k - c(k) \\
&= \delta V_f(k,\mu^0,\pi) - c(k)
\end{aligned}$$

where the inequality follows by Equation (24).

3. Suppose $\mathcal{V}_{\underline{f}}(\mu^0) = \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi$, i.e. $\delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi \geq \max\{\delta\pi, \delta\overline{\mathcal{S}}(\mu^0)\}$.

   I will demonstrate that $k^*(\mu^0,\pi) = \underline{k}(\mu^0)$, i.e. $\delta V_f(\underline{k}(\mu^0),\mu^0,\pi) - c(\underline{k}(\mu^0)) \geq \delta V_f(k,\mu^0,\pi) - c(k)$ for all $k \geq 0$.

   My first claim is that in this case, $\delta V_f(\underline{k}(\mu^0),\mu^0,\pi) - c(\underline{k}(\mu^0)) = \frac{\delta}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\alpha\eta_0)\underline{k}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi - c(\underline{k}(\mu^0))$. This is equivalent to showing that $\underline{k}(\mu^0) \in [\frac{1-\delta}{1-\beta}\frac{1}{\gamma(\mu^0)}\pi, \frac{1-\delta}{1-\beta}\frac{1}{\eta_0}\pi]$.

   To see $\underline{k}(\mu^0) \leq \frac{1-\delta}{1-\beta}\frac{1}{\eta_0}\pi$, suppose not, i.e. suppose $\pi < \frac{1-\beta}{1-\delta}\eta_0\underline{k}(\mu^0)$. Then,

$$
\begin{aligned}
\delta\overline{\mathcal{S}}(\mu^0) &= \frac{\delta}{1-\delta}(1-\beta)(\mu^0+(1-\mu^0)\eta_0)\overline{k}(\mu^0) - c(\overline{k}(\mu^0)) \\
&> \frac{\delta}{1-\delta}(1-\beta)(\mu^0+(1-\mu^0)\eta_0)\underline{k}(\mu^0) - c(\underline{k}(\mu^0)) \\
&> \frac{\delta}{1-\delta}(1-\beta)(\mu^0+(1-\mu^0)\alpha\eta_0)\underline{k}(\mu^0) - c(\underline{k}(\mu^0)) + \delta(1-\mu^0)(1-\alpha)\pi \\
&= \delta\overline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi
\end{aligned}
$$

where the first inequality follows by Equation (24). This contradicts the case considered here.

To see $\underline{k}(\mu^0) \geq \frac{1-\delta}{1-\beta}\frac{1}{\gamma(\mu^0)}\pi$, suppose not, i.e. suppose $\pi > \frac{1-\beta}{1-\delta}\gamma(\mu^0)\underline{k}(\mu^0)$. Then,

$$
\begin{aligned}
\delta\pi &> \frac{\delta}{1-\delta}(1-\beta)\underline{k}(\mu^0)\gamma(\mu^0) \\
&= \frac{\delta}{1-\delta}(1-\beta)\underline{k}(\mu^0)\frac{\mu^0(1-\eta_0)+\eta_0(1-\delta)}{\mu^0(1-\eta_0)\delta+(1-\delta)} \\
&= \frac{\delta}{1-\delta}(1-\beta)\underline{k}(\mu^0)\frac{\mu^0+(1-\mu^0)\alpha\eta_0}{1-(1-\mu^0)(1-\alpha)} \\
&= \frac{\delta}{1-\delta}(1-\beta)(\mu^0+(1-\mu^0)\alpha\eta_0)\underline{k}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi \\
&> \frac{\delta}{1-\delta}(1-\beta)(\mu^0+(1-\mu^0)\alpha\eta_0)\underline{k}(\mu^0) - c(\underline{k}(\mu^0)) + \delta(1-\mu^0)(1-\alpha)\pi \\
&= \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi
\end{aligned}
$$

which contradicts the case considered here. Therefore,

$$
\begin{aligned}
\delta V_f(\underline{k}(\mu^0),\mu^0,\pi) - c(\underline{k}(\mu^0)) &= \frac{\delta}{1-\delta}(1-\beta)(\mu^0+(1-\mu^0)\alpha\eta_0)\underline{k}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi - c(\underline{k}(\mu^0)) \\
&= \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi
\end{aligned}
$$

- For $k \in [0, \frac{1-\delta}{1-\beta}\frac{1}{\gamma(\mu^0)}\pi]$,

$$
\begin{aligned}
\delta V_f(\underline{k}(\mu^0),\mu^0,\pi) - c(\underline{k}(\mu^0)) &= \delta\underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi \\
&\geq \delta\pi \\
&\geq \delta\pi - c(k) \\
&= \delta V_f(k,\mu^0,\pi) - c(k)
\end{aligned}
$$

  where the first inequality holds in the case considered here, and the second inequality holds by Assumption 1.

- For $k \in \left[\frac{1-\delta}{1-\beta}\frac{1}{\gamma(\mu^0)}\pi, \frac{1-\delta}{1-\beta}\frac{1}{\eta_0}\pi\right]$,

$$\begin{aligned}
\delta V_f(\underline{k}(\mu^0), \mu^0, \pi) - c(\underline{k}(\mu^0)) &= \delta \underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi \\
&= \frac{\delta}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\alpha\eta_0)\underline{k}(\mu^0) - c(\underline{k}(\mu^0)) + \delta(1-\mu^0)(1-\alpha)\pi \\
&\geq \frac{\delta}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\alpha\eta_0)k - c(k) + \delta(1-\mu^0)(1-\alpha)\pi \\
&= \delta V_f(k, \mu^0, \pi) - c(k)
\end{aligned}$$

  where the inequality follows by Equation (26).

- For $k \geq \frac{1-\delta}{1-\beta}\frac{1}{\eta_0}\pi$,

$$\begin{aligned}
\delta V_f(\overline{k}(\mu^0), \mu^0, \pi) - c(\overline{k}(\mu^0)) &= \delta \underline{\mathcal{S}}(\mu^0) + \delta(1-\mu^0)(1-\alpha)\pi \\
&\geq \delta \overline{\mathcal{S}}(\mu^0) \\
&= \delta \left(\frac{1}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\eta_0)\overline{k}(\mu^0) - \frac{c(\overline{k}(\mu^0))}{\delta}\right) \\
&\geq \delta \left(\frac{1}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\eta_0)k - \frac{c(k)}{\delta}\right) \\
&= \frac{\delta}{1-\delta}(1-\beta)(\mu^0 + (1-\mu^0)\eta_0)k - c(k) \\
&= \delta V_f(k, \mu^0, \pi) - c(k)
\end{aligned}$$

  where the first inequality holds in the case considered here, and the second inequality follows by Equation (24).

□